Matemáticas VII

Preparaduría

Semana #03.

* CONVOLUCIÓN

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,\partial y$$

a) $f \wedge g$ son causales $\Rightarrow (f*g)(x)\ \exists \wedge$ es causal.

b) Si $f, g \in L^1(\mathbb{R}) \Rightarrow (f*g)(x)\ \exists \wedge \in L^1(\mathbb{R})$.

→ Propiedades.

- $(f*g)(x) = (g*f)(x)$
- $((f*g)(x) * h)(x) = (f*(g*h)(x))(x)$
- $(f*(g+h)(x))(x) = (f*g)(x) + (f*h)(x)$
- $((\alpha f)*g)(x) = (f*(\alpha g))(x) = \alpha (f*g)(x)$
- $(\delta(x) * F(x))(x) = F(x)$ ó $(\delta(x-a) * F(x))(x) = F(x-a)$
- $(F*G)'_{gen} = (F'_{gen}) * G = F * (G'_{gen})$

ó

- $\boxed{(F*G)^{(n+k)}_{gen} = (F^{(n)}_{gen}) * (G^{(k)}_{gen})}$

En sentido distribucional: Si $F \wedge G \in \mathcal{D}(V)$ definimos $F*G \in \mathcal{D}(V)$

$$\langle (F*G)(x) | \varphi(x) \rangle = \langle F(x) | \langle G(x) | \varphi(y-x) \rangle \rangle \; ; \; \forall \varphi$$

* TRANSFORMADA DE LAPLACE

$f: \mathbb{R} \to \mathbb{C}$ Definición $F(z) = \mathcal{L}(f(x))(z) = \int_{-\infty}^{\infty} f(x)\, e^{-zx}\, dx = \langle f | e^{-zx} \rangle(x)$

→ Propiedades.

- $u(x) \xrightarrow{\mathcal{L}} U(z)$
- $u'_{gen}(x) \xrightarrow{\mathcal{L}} z\,U(z)$
- $x\,u(x) \xrightarrow{\mathcal{L}} -U'(z)$
- $u(x-a) \xrightarrow{\mathcal{L}} u(z)\,e^{-az}$
- $e^{\alpha x} u(x) \xrightarrow{\mathcal{L}} U(z-\alpha)$
- $\alpha u(x) + \beta v(x) \xrightarrow{\mathcal{L}} \alpha U(z) + \beta V(z)$
- $(u*v)(x) \xrightarrow{\mathcal{L}} U(z)\,V(z)$

En general.

- $f^{(n)}_{(x)} \xrightarrow{\mathcal{L}} z^n F(z)$
- $x^{(n)} f(x) \xrightarrow{\mathcal{L}} (-1)^n \dfrac{\partial^n}{\partial z^n}(F(z))$

Ejercicio

1) Repaso.

Parcial Enero – Marzo 2007 (10 Ptos).

Sea $f(x)$ definida por:

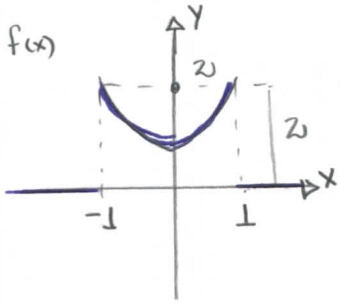
$$f(x) = \begin{cases} x^2+1 & \text{sí } |x| \leq 1 \\ 0 & \text{sí } |x| > 1 \end{cases}$$

a) Calcular f'_{gen}, f''_{gen} y f'''_{gen}

b) Diga sí existe $f(x) * f(x)$ (justifique su respuesta)

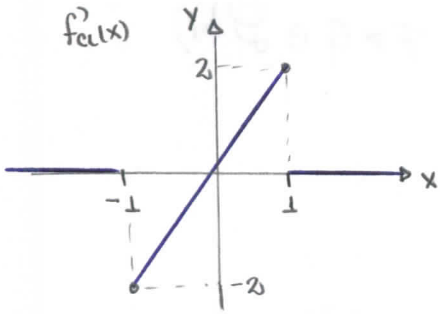
Solución

a)



$$f'_{cl}(x) = \begin{cases} 2x \;; & \text{sí } |x| \leq 1 \\ 0 \;; & \text{sí } |x| > 1 \end{cases}$$

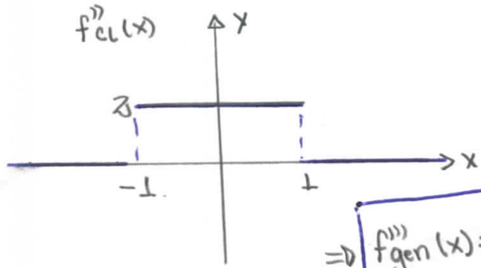
$$\boxed{f'_{gen}(x) = f'_{cl}(x) + 2\delta(x+1) - 2\delta(x-1)}$$



$$f''_{cl}(x) = \begin{cases} 2 \;: & \text{sí } |x| \leq 1 \\ 0 \;; & \text{sí } |x| > 1 \end{cases}$$

$$\boxed{f''_{gen}(x) = f''_{cl}(x) + 2\delta'(x+1) - 2\delta'(x-1) - 2\delta(x+1) - 2\delta(x-1)}$$

$$f'''_{cl}(x) \equiv 0 \;\forall x.$$



$$f'''_{gen}(x) = f'''_{cl}(x) + 2\delta''(x+1) - 2\delta''(x-1) - 2\delta'(x+1) - 2\delta'(x-1) + 2\delta(x+1) - 2\delta(x-1)$$

$$\Rightarrow \boxed{f'''_{gen}(x) = 2\delta''(x+1) - 2\delta''(x-1) - 2\delta'(x+1) - 2\delta'(x-1) + 2\delta(x+1) - 2\delta(x-1)}$$

b) Si existe porque la función es una función causal.

Ejercicios

2) Parcial Sep - Dic 2007 (6 Ptos)

$(\mathbb{1}_{[-1;1]}(x) * \mathbb{1}_{[-1;1]}(x))$. Calcular

Solución Llamamos $u(x) = (\mathbb{1}_{[-1;1]}(x) * \mathbb{1}_{[-1;1]}(x))$

Consideramos $u'_{gen}(x) = (\mathbb{1}_{[-1;1]}(x))'_{gen} * \mathbb{1}_{[-1;1]}(x)$

$$\Rightarrow u'_{gen}(x) = [\delta(x+1) - \delta(x-1)] * \mathbb{1}_{[-1;1]}(x)$$

$$= [\delta(x+1) * \mathbb{1}_{[-1;1]}(x)]_{(x)} - [\delta(x-1) * \mathbb{1}_{[-1;1]}(x)]$$

$$= \mathbb{1}_{[-1;1]}(x+1) - \mathbb{1}_{[-1;1]}(x-1) = \mathbb{1}_{[-2;0]}(x) - \mathbb{1}_{[0;2]}(x)$$

$$\Rightarrow u'_{gen}(x) = \mathbb{1}_{[-2;0]}(x) - \mathbb{1}_{[0;2]}(x)$$

$$\mathbb{1}_{[-2;0]}(x) = H(x+2) - H(x) \quad \wedge \quad \mathbb{1}_{[0;2]}(x) = H(x) - H(x-2)$$

$$\Rightarrow \boxed{u'_{gen}(x) = H(x+2) - 2H(x) + H(x-2)}$$

Integramos

$$u(x) = (x+2)H(x+2) - 2xH(x) + (x-2)H(x-2) + C$$

La función debe ser causal por lo tanto $c \equiv 0$

$$\Rightarrow \boxed{u(x) = (x+2)H(x+2) - 2xH(x) + (x-2)H(x-2)}$$

3). Parcial Enero - Marzo 2012. (10 Ptos)

Sean $f(x)$ y $P(x)$ dada por.

$$f(x)=\begin{cases} 2+x \,;\; x\in[-2;0] \\ 2-x \,;\; x\in[0;2] \\ 0 \text{ en cualquier otro } x \end{cases} \qquad P(x)=\begin{cases} 1 \,;\; |x|<1 \\ 0 \,;\; |x|>1 \end{cases}$$

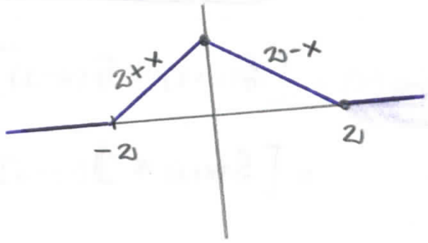
Sea $K=f*P$.

a) Calcule $K'_{gen}(x)$

b) Calcule K.

c) Calcule la transformada de Laplace de $K'_{gen}(x)$

$f(x) \longrightarrow$



Solución (MÉTODO #01)

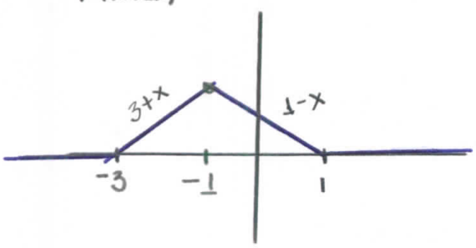
$K=P*f \Rightarrow K'_{gen} = P'_{gen} * f$

$$P(x)=\begin{cases} 1 \,;\; |x|<1 \\ 0 \,;\; |x|>1 \end{cases} \Rightarrow \boxed{P'_{gen} = \delta(x+1)-\delta(x-1)}$$

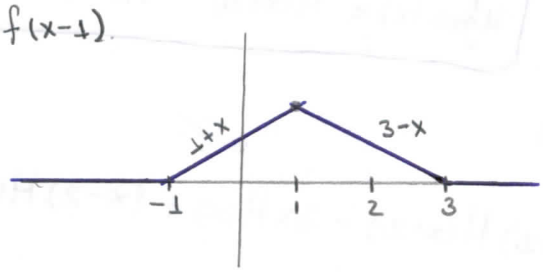
$$\Rightarrow K'_{gen} = [\delta(x+1)-\delta(x-1)]*f = (\delta(x+1)*f)-(\delta(x-1)*f)$$

$$= f(x+1)-f(x-1)$$

$f(x+1)$



$f(x-1)$



$$\Rightarrow K'_{gen} = f(x+1)-f(x-1)$$

$$\Rightarrow K'_{gen}(x) = (3+x)\big(H(x+3)-H(x+1)\big)+(1-x)\big(H(x+1)-H(x-1)\big)-(1+x)\big(H(x+1)-H(x-1)\big)-(3-x)\big(H(x-1)-H(x-3)\big)$$

$$\Rightarrow K(x) = \frac{(3+x)^2}{2}\big(H(x+3)-H(x+1)\big)+\frac{(1-x)^2}{2}\big(H(x+1)-H(x-1)\big)-\frac{(1+x)^2}{2}\big(H(x+1)-H(x-1)\big)-\frac{(3-x)^2}{2}\big(H(x-1)-H(x-3)\big)+C$$

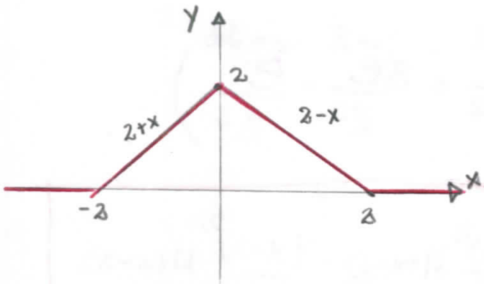
MÉTODO #02.

$$\boxed{K=(f*P)_{(x)}=(P*f)_{(x)}}$$

$$\boxed{(f*P)_{(x)}\xrightarrow{\mathcal{L}}F(z)P(z)}\ \text{(I)} \quad \wedge \quad \boxed{F(z)P(z)\xrightarrow{\mathcal{L}^{-1}}(f*P)(x)}\ \text{(II)}.$$

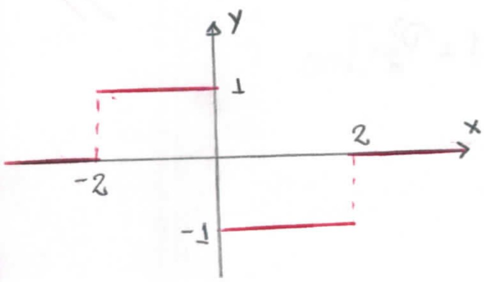
a)

$$f(x)=\begin{cases}2+x\ ; & x\in[-2;0]\\ 2-x\ ; & x\in[0;2]\\ 0\ ; & \text{En cualquier otro } x.\end{cases}$$



$$f'_{gen}(x)=f'_{CL}(x)=\begin{cases}1 & x\in[-2;0]\\ -1 & x\in[0;2]\\ 0 & x\notin[-2;2]\end{cases}$$

$$\boxed{f'_{gen}(x)=f'_{CL}(x)} \Rightarrow \boxed{f''_{CL}(x)\equiv 0}$$



$$f''_{gen}(x)=\underbrace{f''_{CL}(x)}_{=0}+\delta(x+2)-2\delta(x)+\delta 2(x-2)$$

$$\boxed{f''_{gen}(x)=\delta(x+2)-2\delta(x)+\delta(x-2)}$$

TABLA : $\boxed{f''_{gen}(x)\xrightarrow{\mathcal{L}}z^2F(z)}$

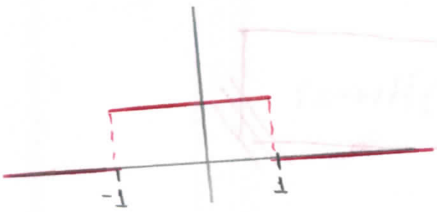
$$\delta(x+2)-2\delta(x)+\delta(x-2)\xrightarrow{\mathcal{L}}e^{2z}-2+e^{-2z}\Rightarrow z^2F(z)=e^{2z}-2+e^{-2z}$$

$$\Rightarrow \boxed{F(z)=\frac{e^{2z}-2+e^{-2z}}{z^2}}$$

$$P(x)=\begin{cases}1\ ; & |x|\leq 1\\ 0\ ; & |x|>1.\end{cases}$$

$$P'_{gen}(x)=\underbrace{P'_{CL}(x)}_{=0}+\delta(x+1)-\delta(x-1) \Rightarrow \boxed{P'_{gen}(x)=\delta(x+1)-\delta(x-1)}$$

TABLA: $\boxed{P'_{gen}(x)\xrightarrow{\mathcal{L}}zP(z)}$



$$\delta(x+1)-\delta(x-1)\xrightarrow{\mathcal{L}}e^{z}-e^{-z}\Rightarrow \boxed{zP(z)=e^{z}-e^{-z}}$$

$$\Rightarrow \boxed{P(z)=\frac{e^{z}-e^{-z}}{z}}$$

$$\text{I} \ldots \text{2} \triangleright \ F(z)P(z) = \left(\frac{e^{2z}+e^{-2z}-2}{z^2}\right)\left(\frac{e^{z}-e^{-z}}{z}\right)$$

$$= \frac{1}{z^3}\left[e^{3z}+e^{-z}-2e^{z}-e^{z}-e^{-3z}+2e^{-z}\right]$$

$$\Rightarrow \boxed{F(z)P(z) = \frac{e^{3z}}{z^3} - \frac{3e^{z}}{z^3} + \frac{3e^{-z}}{z^3} - \frac{e^{-3z}}{z^3}}$$

$$\text{II} \ldots \text{2} \triangleright \ F(z)P(z) \xrightarrow{\mathcal{L}^{-1}} (f*P)_{(x)} \Rightarrow \mathcal{L}^{-1}\left(\frac{e^{3z}}{z^3} - \frac{3e^{z}}{z^3} + \frac{3e^{-z}}{z^3} - \frac{e^{-3z}}{z^3}\right)$$

$$\Rightarrow \boxed{(f*P)_{(x)} = K(x) = \frac{(x+3)^2}{2!}H(x+3) - 3\frac{(x+1)^2}{2!}H(x+1) + 3\frac{(x-1)^2}{2!}H(x-1) - \frac{(x-3)^2}{2!}H(x-3).}$$

b) $K'_{gen}(x) = (f*P)'_{gen}(x) = (P*f)'_{gen}(x) = \left[f'_{gen}*P\right]_{(x)} = \left[f*P'_{gen}\right]_{(x)}$.

$$\left[f*(P'_{gen})\right]_{(x)} \xrightarrow{\mathcal{L}} F(z)zP(z).$$

$$\Rightarrow F(z)zP(z) = \left[\frac{e^{2z}+e^{-2z}-2}{z^2}\right]\left[\frac{e^{z}-e^{-z}}{\cancel{z}}\right](\cancel{z})$$

$$\Rightarrow \boxed{\mathcal{L}(K'_{gen}(x))(z) = \frac{1}{z^2}\left(e^{3z}-3e^{z}+3e^{-z}-e^{-3z}\right)}$$

c) $\mathcal{L}^{-1}\left(\mathcal{L}(K'_{gen}(x))(z)\right)_{(x)} = K'_{gen}(x)$

$$\Rightarrow K'_{gen}(x) = \mathcal{L}^{-1}\left(\frac{e^{3z}}{z^2} - \frac{3e^{z}}{z^2} + \frac{3e^{-z}}{z^2} - \frac{e^{-3z}}{z^2}\right)_{(x)}$$

$$\Rightarrow \boxed{K'_{gen}(x) = (x+3)H(x+3) - 3(x+1)H(x+1) + 3(x-1)H(x-1) - (x-3)H(x-3)}$$

4) Calcule la función $u(x)$ cuya transformada viene dada. (6 Ptos)

Parcial

Enero - Marzo 2008.

$$\mathcal{L}(u(x))(z) = U(z) = \left(\frac{10-3z^2+z^3}{z^2+1}\right)e^{8z}$$

Solución

Consideramos $\dfrac{10-3z^2+z^3}{z^2+1}$, por el momento.

$$\frac{10-3z^2+z^3}{z^2+1} = z-3+\frac{13-z}{z^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{10-3z^2+z^3}{z^2+1}\right)(x) = \mathcal{L}^{-1}(z-3)(x) - \mathcal{L}^{-1}\left(\frac{z}{z^2+1}\right) + 13\,\mathcal{L}^{-1}\left(\frac{1}{z^2+1}\right).$$

$$= \delta'(x) - 3\delta(x) - \cos(x)H(x) + 13\,\mathrm{sen}(x)H(x).$$

Recordando $U(z)e^{-az} \xrightarrow{\mathcal{L}^{-1}} u(x-a)$; $e^{8z} \Rightarrow \boxed{a=-8}$

$$\Rightarrow \boxed{u(x) = \delta'(x+8) - 3\delta(x+8) - \cos(x+8)H(x+8) + 13\,\mathrm{sen}(x+8)H(x+8)}$$

5) Parcial Enero - Marzo 2009. (35%) (3-4). Torno.

(8 Ptos) Sea $f(x) = \begin{cases} 1 & \text{sí } -1<x<1 \\ 0 & \text{sí } x\notin[-1;1] \end{cases}$

Calcule el producto de convolución $f*f$.

Solución

$f(x) \xrightarrow{\mathcal{L}} F(z) = \dfrac{e^{z}}{z} - \dfrac{e^{-z}}{z} \Rightarrow \boxed{F(z) = \left(e^{z}-e^{-z}\right)\frac{1}{z}}$

$f(x) = H(x+1) - H(x-1)$

$$(f*f)(x) \xrightarrow{\mathcal{L}} F(z)\cdot F(z) = (F(z))^2$$

$$(F(z))^2 = \frac{(e^{z})^2 - 2(e^{z})(e^{-z}) + (e^{-z})^2}{z^2} \Rightarrow (F(z))^2 = \frac{e^{2z}}{z^2} - \frac{2}{z^2} + \frac{e^{-2z}}{z^2}$$

$$(F(z))^2 \xrightarrow{\mathcal{L}^{-1}} (f*f)(x) \Longrightarrow \mathcal{L}^{-1}\left(\frac{e^{2z}}{z^2} - \frac{2}{z^2} + \frac{e^{-2z}}{z^2}\right)$$

$$\Rightarrow \boxed{(f*f)(x) = (x+2)H(x+2) - 2xH(x) + (x-2)H(x-2)}$$